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A Single Period Inventory Problem with Quadratic Demand Distribution under the Influence of Marketing Policies

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Abstract. A single-period inventory problem is considered where demand is taken to be quadratic in nature. The model contains usual ingredients of a newsboy problem: sale price, purchase cost, salvage value and shortage cost. In addition, there is a cost per unit of marketing effort, and the demand distribution is stochastically increasing in that effort. It is assumed that the distribution of demand can be shifted up by an increase in sales effort. The paper deals with the simultaneous determination of optimal order quantity and sales effort. The optimum profit associated with optimum marketing effort is also determined. The sensitivity analysis of the results is provided, to consider variation in optimal order quantity and sales effort with the changes in unit sales price, unit purchase cost, unit salvage value, unit shortage cost and cost per unit of marketing effort.

1. Introduction

The single-period inventory problem can be summarized as follows: A single order can be placed for delivery of an item before the beginning of a period, which represents the duration of the same planning horizon. There is either no opportunity of placing any subsequent orders during the period, or there is a penalty cost per item for special orders placed during the period. Items remaining at the end of the period are disposed off at the same (possibly negative) value, and unsatisfied demand results in a penalty cost. Such problems are associated with the inventory of an item having one or more of the following characteristics:

a) Becomes obsolete quickly, *e.g.* newspapers, fashion goods etc.
b) Spoils quickly, *e.g.* fruit, vegetable etc.

c) Seasonal goods where a second order during the season is difficult.
d) Stocked only once, *e.g.* spare parts for a single production run of products.
e) Has a future that is uncertain beyond the planning horizon.

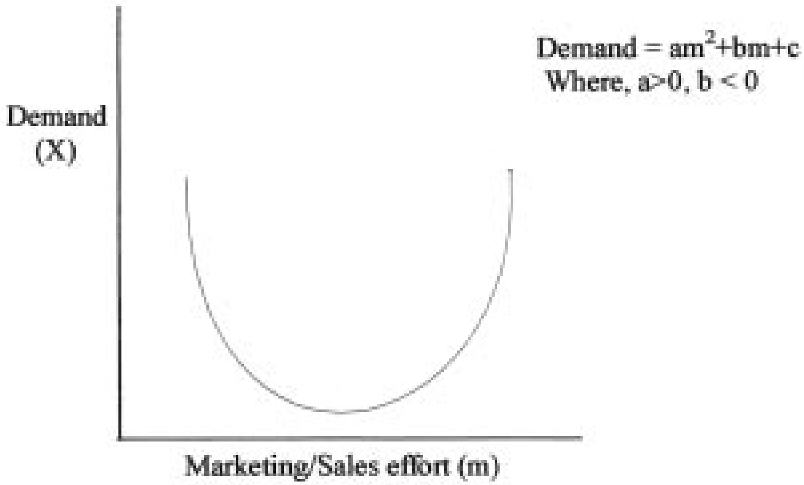
Inventory theory typically depicts demand distribution as exogenous to the models. For the firm as a whole, however, demand can obviously be affected by pricing decisions, promotions and service quality. In turn, inventory policies themselves can occasionally affect demand, as in the case of stockout frequency. Eillon[1], Sengupta[2] has discussed some joint pricing and inventory decisions in an oligopoly situation. Lal and Staelin[3] derived quantity-discount pricing policies which reduce the joint (buyer and seller) ordering and holding costs. Balcer[4,5] proposed a dynamic model for optimal advertising and inventory control. The EOQ has been identified and studied by Kotler[6] and by Ladaney and Sternlieb[8], who have considered the effect of price variation on demand with the objective of profit maximization. The studies of Kotler[6] and Ladaney[8] deal with the objective of profit maximization. The studies of Kotler[6] and Ladaney[8] deal with deterministic situations. A single period stochastic inventory model under influence of marketing policies is studied by Shah and Jha[9]. Shore[10] considered approximations for the demand distribution in single-period inventory problem. Jucker and Rosenblatt[11] considered quantity discount schemes. A single period inventory problem with partially controllable demand is given by Gerchak and Parlar[12] and with triangular demand distribution by Walker[13]. The optimization model for customer redistribution is studied by Sydney[14].

This paper addresses the simultaneous determination of order quantity and sales effort in the simplest scenario, that of a single period 'newsboy' problem (*e.g.* Silver and Peterson[7], the vendor is assumed to be able to shift the demand distribution up by an increase in sales effort.

A single-period inventory problem with quadratic demand distribution, under the influence of marking policies is considered. It is assumed that the distribution of demand can be shifted up by an increase in sales effort. The paper deals with the simultaneous determination of optimal order quantity and sales effort. The optimum profit associated with optimum marketing effort is determined. The sensitivity analysis results are discussed, to consider variation in optimal order quantity and sales effort with the changes in key parameters used in the model.

2. Mathematical Model

In this model we consider demand to be quadratic in nature, which is justified as follows:



An innovative product, says a new model of leather shoe, is launched specially for winter season. The increase in marketing effort, upto a certain level may not lead to an increase in demand due to competition from traditional well accepted products. However, the company may be able to increase the demand in an appreciable manner with innovative advertising and sales policy which offer complete satisfaction to customers.

In this model the random demand X for the product is a function of the potential market of size N which is a random variable with known probability density $g_N(n)$. The potential market can be reached with infinitely large marketing (sales) effort, and for a finite level of effort 'm', the demand X is a product of N and the diminishing returns on effort term $[am^2 + bm + c]$ *i.e.*

$$X = [am^2 + bm + c]\, N \tag{1}$$

If no special sales effort is undertaken (m = 0), then Eq. (1) reduces to X = CN

As in the classical newsboy problem we define the following parameters.

S = Unit sale price
C = Unit purchase cost (C < S)
V = Unit salvage value
P = Unit shortage cost
and, r = Cost per unit of marketing effort.

Now, defining q to be the order quantity and π the random profit we have

$$\pi(q,\ m\,|\,x) = \begin{cases} Sx + v(q-x) - cq - rm, & x < q \\ Sq - p(x-q) - cq - rm & x \geq q \end{cases} \tag{2}$$

Here, given the realized value X = x of the demand, the profit is calculated as a function of the two decision variables viz. the order quantity ‘q’ the marketing effort ‘m’. As in the classical model, the problem is to find the optimal value of the decision variables in order to maximize the expected profit $E\pi = f(q, m)$.

Recalling the $X = [am^2 + bm + c] N$, we note that for an observed value of N = n, $X = [am^2 + bm + c] n$, substituting this into the Eq. (2) and integrating we obtain

$$E\pi = f(q,m) = \int_0^L S(am^2 + bm + c)n + v[q - (am^2 + bm + c)n]\} \; g(n) \; dn \tag{3}$$

$$+\int_L^\infty \{Sq - p[(am^2 + bm + c)n - q]\} \; g(n) \; dn - cq - rm \tag{4}$$

where, $L = L(q, m) = q / (am^2 + bm + c)$

The problem, therefore, is to find the optimal nonnegative values of the order quantity and marketing effort level ‘q’ and ‘m’, to maximize the expected profit $E\pi = f(q, m)$, *i.e.*

$$\text{Maximize} \quad f(q, m) \qquad 4(a)$$

$$\text{Subject to} \quad q \geq 0, m \geq 0 \qquad 4(b)$$

The concavity property of the objective function over the nonnegative quadrant $Q = \{(q, m) \mid q \geq 0, m \geq 0\}$ is analyzed as given below:

Forming the Lagrangian $\Lambda = f(q, m) + \mu_1 q + \mu_2 m$, we obtain the Kuhn-Tucker necessary conditions a

$$\partial\Lambda / \partial q = f_q + \mu_1 = 0 \qquad 5(a)$$

$$\partial\Lambda / \partial m = f_m + \mu_2 = 0 \qquad 5(b)$$

$$\mu_1 q = 0 \; ; \quad \mu_1 \geq 0 \qquad 5(c)$$

$$\mu_2 m = 0 \; ; \quad \mu_2 \geq 0 \qquad 5(d)$$

where,

$$f_q = (v - c) + (s + p - v) \int_L^\infty g(n) \; dn \tag{6}$$

$$f_m = (s - v)(2am + b) \; \overline{N} - (s + p - v) \; (2am + b) \int_L^\infty ng(n) \; dn - r \tag{7}$$

Where $\overline{N} = \int_L^\infty ng(n) \; dn$ is the expected market potential reachable with infinitely large marketing effort.

We note that when m = 0 the problem reduces to the classical model for which necessary conditions is $f_q = 0$.

Therefore from Eq. (6) we have

$$(6) \Rightarrow \int_L^\infty g(n)\ dn = (c - v)/(s - v + p) \ \text{ as expected} \tag{8}$$

we now examine the behaviour of the necessary conditions at the boundaries and in the interior of the first quadrant Q

Case 1

$$(q = 0, m \geq 0)$$

When q = 0, Eq. 5(b) gives, $f_m + \mu_2 = 0$. equivalently,

$(s - v)(2am + b)\bar{N} - (s + p - v)(2am + b)\int_L^\infty ng(n)dn - r + \mu_2 = 0$. Observed that L = 0 and

$$\mu_2 = p(2am + b)\ \bar{N} r \geq 0 \tag{9}$$

Which in turn implies m = 0 by Eq. 5(d). The interpretation of this result is: when there is nothing ordered, the marketing effort should also be zero. This, of course means that $f(0, 0) = - p(c)\bar{N} < 0$ indicating that q = m = 0 cannot be an optional solution if p > 0 since the expected profit would be negative due to the penalty resulting from the expected unsatisfied demand of $(c)\bar{N}$ units.

Case 2

$$(q \geq 0, m = 0)$$

When m = 0 then L = q/c which when combined with Eq. 5(a) gives

$$f_q + \mu_1 = 0$$

$$(v - c) + (s + p - v) \int_L^\infty g(n)\ dn + \mu_1 = 0$$

Or

$$\mu_1 = (c - v) - (s + p - v)\int_L^\infty g(n)\ dn \tag{10}$$

It is now demonstrated (by contradiction) that $\mu_1 = 0$ and q > 0: assume $\mu_1 >$ 0, then q = 0 from Eq. 5(c), and L = 0. This implies, the property of random variable.

$\int_{L}^{\infty} g(n)\ dn = \int_{0}^{\infty} g(n)\ dn = 1,$ and from Eq. (10) we have.

$$(10) \Rightarrow \ \mu_1 = (c - v) - (s + p - v) = (c - s - p) > 0$$

or

$$c > (s + p)$$

Which is a contradiction since the unit purchase cost 'c' is less than unit price 's'. Therefore it is concluded that when there is no marketing effort (m = 0), the optimal value of the order quantity 'q' cannot be zero. It must be nonnegative. In this case q^* is found from Eq. (10) as

$$\int_{q^*/c}^{\infty} g(n)\ dn = (c - v)/(s + p - v) \tag{11}$$

Which is the solution to the classical newsboy with the demand given by X = (1 – b)N

Case 3

$$(q > 0\ 0,\ m > 0)$$

In this case, the necessary conditions Eq. 5(c) and Eq. 5(d) imply $\mu_1 = \mu_2 = 0$ and the optimal solution is found by solving $f_q = f_m = 0$.

The conclusion reached by the analysis of the above three cases is that the optimal solution $(q^*, m^*) \in Q = \{q, m) \mid q > 0, m \geq 0\}$. The concavity of the objective function which indicates uniqueness of the solution will be the next stage in our analysis of the necessary condition for this problem.

Theorem

There is a unique optimal solution $(q^*, m^*) \in Q$

Proof

To show the uniqueness of the solution it is demonstrated that the expected profit function f(q, m) is concave on Q, or (i) $f_{qq} < 0$ and (ii) $|H| < 0$ where

$$H = \begin{Bmatrix} f_{qq} & f_{qm} \\ f_{mq} & f_{mm} \end{Bmatrix}$$

is the Hessian matrix, and |H| is its determinant

By repeated application of the Leibnitz's rule for differentiation under integral sign

$$f_{qq} = -\, g(L)\,(s + p - v) \qquad 1 / (am^2 + bm + c) \text{ was obtained} \tag{12}$$

$$f_{qm} = f_{mq} = [g(L)\,(s + p - v)\, q(2am + b)] / (am^2 + bm + c)^2 \tag{13}$$

Notice that $f_m = (s - v)\,(2am + b)\,\overline{N} - (s + p - v)\,(2am + b) \int_L^\infty ng(n)\ dn - r$, a

$$f_{mm} = (s - v)\,2a\,\overline{N} - (s + p - v)\,[2a\int_L^\infty ng(n)\,dn + Lg\,(L)\,\{q(2am + b)^2 / (am^2 + bm + c)^2\}] \tag{14}$$

Where as before $L = q / (am^2 + bm + c)$. The determinant of Hessian is

$$|H| = f_{qq}\, f_{mm} - (f_{qm})^2$$

using Eq. (12), Eq. (13) and Eq. (14), then

$$|\,H\,| = [(s + p - v)\, g(L) / (am^2 + bm + c)]\ [-(s - v)\, 2a\overline{N} + (s + p - v)\,\{2a\int_L^\infty ng(n)\,dn$$

$$+ L\, g(L)\, q(2am + b)^2 / (am^2 + bm + c)^2\,\}$$

$$-\, g(L)\, q^2(2am + b)^2\,(s + p - v)\,(am^2 + bm + c)^3] \tag{15}$$

It is noted that although $f_{qq} < 0$, the sign of $|H|$ is not immediately obvious since the second term can be positive or negative depending on the value of $L = L\,(q, m)$. But, recalling the necessary condition $f_m = 0$ for an interior point (q, m) with $q > 0$, $m > 0$. It was noted that

$$f_m = (s - v)\,(2am + b)\,\overline{N} - (s + p - v)\,(2am + b) \int_L^\infty ng(n)\ dn - r = 0, \text{ which implies}$$

$$(s - v)\,\overline{N} - (s + p - v) \int_L^\infty ng(n)\,dn = r / (2am + b) > 0 \tag{16}$$

By using Eq. (16) in Eq. (15), then

$$|H| = -[2a(s + p - v)\ \ g(L) / (am^2 + bm + c)]\ \ [r / 2am + b) \quad < 0$$

Resulting in a negative value of $|H|$. Therefore, if there exists an interior point (q, m), then the function $f(q, m)$ takes on a local maximum value at that point since $f_{qq} < 0$ and $|H| < 0$, but if all the interior stationary points are local maxima then there must be a unique interior maximum (q^*, m^*) $\{ (q, m) \mid q > 0, m > 0 \}$ otherwise the maximum is at the boundary given by

$$\int_{q^*/c}^\infty g(n)\,dn = (c - v) / (s + p - v) \tag{17(a)}$$

$$m^* = 0 \tag{17(b)}$$

as in the classical newsboy problem **Q.E.D.**

As an example, let us consider the special case were N is exponential *i.e.* g(n) = $\lambda_e^{-\lambda n}$ n ≥ 0. Using the necessary conditions then the following interior optimal solution is obtained:

$$q^* = [(\log Z) / \lambda] \ (am^2 + bm + c) \tag{18}$$

$$m^* = [1/2a] \ [r\,\lambda / \{ (s - c) - (c - v) \ \log Z \} - b] \tag{19}$$

where Z = (s + p – v) / (c – v)

If for some values of the parameters, m^* is negative then $m^* = 0$. Substituting m^* in Eq. (18) gives the optimum order quantity. The optimum profit is given by Eq. (3)

$$f^* = \int_0^\infty \{ s\,(am^2 + bm + c)\,n + v\,[q - (am^2 + bm + c)\,n]\}\,g(n)\,dn$$

$$+ \int_L^\infty \{ sq - p\,[(am^2 + bm + c)\,n - q]\}\,g(n)\,dn - cq - rm$$

putting $g(n) = \lambda e^{-\lambda n}$, m^* and q^*

$$f^* = [(am^{*2} + bm^* + c) \ (s - v) / \lambda + (v - c)\,q^* - rm^* -$$

$$[e^{-\lambda q^*/(am^{*2} + bm^* + c)} \ (am^* + bm^* = c)\ (s - v + p)] / \lambda$$

$$f^* = T(s - v) / \lambda + (v - c)\,q^* - rm - [e^{-\lambda q^*/T} \ T(s - v + p)] / \lambda \tag{20}$$

where $T = am^{*2} + bm^* + c$

The numerical example is illustrated to explain quadratic nature of demand.

3. Numerical Results and Discussions

In this section the solutions of numerical example for different parameter-values is presented. The optimal order quantity q^* and marketing effort m^* values are compared for a range of the parameters.

The following reference values were used which were varied to obtain the sensitivity analysis results for the optimal values of the decision variables q and m:

S = \$12 C = \$10 P = \$4 V = \$7

r = \$5 $\lambda = 0.1$ $\bar{N} = 10$ a = 2 b = – 8 c = 1

Sensitivity analysis results are presented in Table 1 are for different values of the first five parameters. The 'λ' parameter is fixed at 0.1 so that $\bar{N}$ is the size of the average potential market for every problem.

TABLE 1. Sensitivity analysis table for the model.

S	10	12	14	16
q*	15.9904	30.6045	30.0687	26.45443
m*	1.8868	2.2197	2.0541	2.0279
C	8	9	10	11
q*	50.5058	35.6235	30.6045	11.7995
m*	2.0410	2.0738	2.2197	1.6940
P	0	2	4	6
q*	12.3744	21.6148	30.6045	43.8142
m*	2.0936	2.13946	2.2197	2.4062
V	3	5	7	9
q*	36.4906	27.3251	30.6045	12.3050
m*	2.0936	2.13946	2.2197	2.4062
r	0	5	10	15
q*	23.8855	30.6045	38.3281	46.9912
m*	2	2.2197	2.4395	2.6593

Optimal q^*, m^* values were obtained by using the well-known Newton method, which makes use of the Hessian matrix. As shown in the model, the decision variables are such that either (i) $q^* > 0$, $m^* = 0$ or (ii) $q^* > 0$ and $m^* > 0$. It is noted that when one parameter value (*e.g.* S) is varied the others (*i.e.* c, p, v, r, a, b and λ) are kept at their reference values. The solutions for the reference values are shown in boxes in boxes in Table 1.

As expected, when 'S' increases, the order quantity q^*, and the marketing effort m^* also increase. But after a certain sale price it shows decrease in q^* and m^* which may be attributed to market maouvres in operation. But when 'c' increases, q^* and m^* move in opposite direction to 'c'. An increase is shortage penalty cost increases q^* and m^*.

The salvage value term 'v' has decreasing effect on q^* and m^* respectively. When cost of marketing effort is increased it leads to an increase in m^* and consequently leads to an increase in m^* and consequently leads to an increase in the optimal order quantity q^*. If $r = 0$ it is noted that effort variable is found to be $m^* = 2$ units.

4. Conclusions

The purpose of this study was to incorporate a marketing variable into an inventory control decision. The analysis was done in the context of a single-period inventory problem. The decision variables namely Optimal order quantity and Optimal marketing effort was determined. With the help of Kuhn-Tucker conditions the behaviour of the necessary conditions at the boundaries was examined and in the interior of the first quadrant Q. If $q = 0, m \geq 0$, the conclusion of this result is when there is nothing ordered the marketing effort should be zero. And if ≥ 0, $m = 0$ which leads to the conclusion that when there is no marketing effort $m = 0$, the optimal value of order quantity 'q' cannot be zero and it must be nonnegative. Thus, in addition to the well-known tradeoffs associated with stocking of a single-period product under uncertain demand, there is an additional tradeoff associated with the sales effort. The proposed model and its solution is interpreted and compared for some particular set of parameter values in numerical sensitivity analysis.

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مشكلة الجـرد أحاديـة الفترة مع التوزيـع التربيـعي للطلب في ظـل أثـر سياسات التسـويق

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المستخلص . تم التعرض لمسألة جرد أحادية الفترة حيث اعتبر الطلب تربيعيًا في طبيعته . تم استعمال نموذج يتضمن المكونات التقليدية لمثل هذه المسألة : سعر البيع وتكلفة الشراء وقيمة التخليص وتكلفة العجز أو النقصان . وبالإضافة إلى ذلك أضيفت تكلفة لكل وحدة من جهد التسويق ، واعتبر توزيع الطلب دالة متزايدة عشوائيًا في ذلك الجهد . وجرى افتراض أن توزيع الطلب يمكن زحزحته لأعلى عن طريق زيادة جهد المبيعات . تناولت الورقة مسألة التقرير الآني للقيمتين المثليين لكمية المطلوب وجهد المبيعات . كذلك تم تحديد الربح الأمثل المرتبط بجهد التسويق الأمثل . تم عمل تحليل لحساسية النتائج وذلك باعتبار التغيرات في كمية المطلوب المثلى وجهد المبيعات الناشئة عن التغيرات في سعر البيع للوحدة وتكلفة الشراء للوحدة وقيمة التخليص للوحدة وكلفة العجز للوحدة وكلفة جهد التسويق لكل وحدة .